1 Cones of K3

The $\mathbb{R}$-vector space $\mathrm{NS}(X)_{\mathbb{R}}$ is ρ-dimensional with quadratic form on it of signature $(1, \rho-1)$. The geometry of a K3 (even for any projective variety) is mainly controlled by its various cones. Recall

Definition 1.1. *Let X be a smooth projective surface over $\mathbb{C}$, set*

1. *the positive cone of X to be*
$$\mathrm{Pos}(X) := \{\alpha \in \mathrm{NS}(X)_{\mathbb{R}} | \alpha^2 > 0\}.$$
2. *the ample cone $\mathrm{Amp}(X) \subset \mathrm{NS}(X)_{\mathbb{R}}$ is generated by ample classes (i.e., those $c_1(L)$ for an ample line bundle L on X).*
3. *the nef cone*
$$\mathrm{Nef}(X) := \{\alpha \in \mathrm{NS}(X)_{\mathbb{R}} \mid \alpha.C \geq 0 \textit{ for any irreducible curve } C \subset X\}$$
4. *the effective cone*
$$\mathrm{Eff}(X) := \{\sum_{\textit{finite}} a_i[C_i] \mid C_i \subset X \textit{ effective curve and } a_i \in \mathbb{R}_{>0}\}$$
5. *the Kahler cone $\mathcal{K}_X \subset H^{1,1}(X,\mathbb{R}) := H^2(X,\mathbb{R}) \cap H^1(X, \Omega_X)$ consists of all Kahler classes.*

It is easy to see from the definition that $\mathrm{Pos}(X)$, $\mathrm{Nef}(X)$ and $\mathrm{Amp}(X)$ are all convex cone in $\mathrm{NS}(X)_{\mathbb{R}}$. By the definition, and $\mathrm{Nef}(X)$ is dual to $\mathrm{Eff}(X)$.

Theorem 1.2. *(Nakai-Moishen-Kleiman's ampleness criterion) Let X be a smooth projective surface over $\mathbb{C}$ and $L\,\mathrm{Pic}(X)$, then L is ample if and only if*

$$L^2 > 0, \quad L.C > 0 \textit{ for any irreducible curve } C \subset X.$$

In particular, this shows
$$\mathrm{Amp}(X) = Int(\mathrm{Nef}(X)) \subset \overline{\mathrm{Amp}(X)} = \mathrm{Nef}(X)$$
where taking interior $Int()$ and closure $\overline{()}$ are under the Euclidean topology of $\mathrm{NS}(X)_{\mathbb{R}}$.

Now we turn to the case X is a K3, where the beautiful geometry of K3 gives a much simpler characterization of the ample cone as follows

Theorem 1.3. *Let X be a K3, then*

$$\mathrm{Amp}(X) = \{L \in \mathrm{Pos}(X) \; | L.C > 0 \textit{ for any } C \cong \mathbb{P}^1 \;\} \tag{1}$$

Proof. Note that for an integral curve $C \subset X$ not isomorphic to $\mathbb{P}^1$, then $C^2 \geq 0$. By the Nakai-Moishen-Kleiman's ampleness criterion (1.2), it is sufficient to show if $C \subset X$ is an integral curve with $C^2 \geq 0$, then $L.C > 0$ for any $L \in \mathrm{Pos}(X)$. As C is effective and nontrivial, then $h^0(-C) = 0$ and Riemann-Roch formula implies $h^0(C) = h^1(C) + 2 + \frac{C^2}{2} \geq 2$. ☐

Theorem 1.4 (Kawamata-Morrison's cone conjecture holds for K3)**.** *There is a rational polyhedral subset $II \subset \mathrm{Nef}^e(X)$* [1] *which is a fundamental domain for the action $\mathrm{Aut}(X)$ on $\mathrm{Nef}^e(X)$. That is,*

$$\mathrm{Aut}(X) \cdot II = \mathrm{Nef}^e(X), \qquad g \cdot II \cap II \textit{ is nonempty if and only if } g = Id$$

Let $V = V_{\mathbb{Q}} \otimes \mathbb{R} \cong \mathbb{R}^{n+1}$ a real vector space with $\langle x, y\rangle := x_0y_0 - \sum_{i=1}^{n} x_iy_i$. The example to keep in mind should be $V = (\mathrm{NS}(X)_{\mathbb{R}}, \cup)$ for a smooth projective surface. Under the non-degenerate bilinear form $\langle , \rangle$, we can identify V with its dual vector space $V^* := Hom(V, \mathbb{R})$ via $v \mapsto l_v$ where $l_v : V \to V$ is the linear map defined by $l_v(w) := \langle v, w\rangle$.

Definition 1.5. *Let $\mathcal{C} \subset V$ be a nondegenarate convex cone.*

[1] $\mathrm{Nef}^e(X)$ is the convex hull of $\overline{\mathrm{Nef}(X)} \cap \mathrm{NS}(X)_{\mathbb{Q}}$.

1. *$\mathcal{C} \subset V$ is called homogeneous if the group*

$$\mathrm{Aut}(\mathcal{C}) := \{g \in GL(V) \mid g(\mathcal{C}) = \mathcal{C}\}$$

acts $\mathcal{C}$ transitively.

2. *$\mathcal{C} \subset V$ is called self-dual if $\mathcal{C} = \mathcal{C}^*$ where $\mathcal{C}^* \subset V^*$ is the dual cone defined as the interior of the set*

$$\{l \in V^* \mid l(w) \geq 0 \text{ for any } w \in \mathcal{C}\}.$$

3. *$\mathcal{C}^+ \subset V$ is the smallest convex cone containing all $\mathbb{Q}$-points of $\overline{\mathcal{C}}$. In other words, $\mathcal{C}^+$ is the convex hull of $V_{\mathbb{Q}} \cap \overline{\mathcal{C}}$.*

Due to the work of Ash ([2][chapter 2]), Looijenga ([4]), there is a very useful criterion for the existence of fundamental domain for the action of any arithmetic group Γ on a cone $\mathcal{C} \subset V$.

Lemma 1.6. *Let $\mathcal{C} \subset V$ be a homogeneous self-dual cone, then*

1. *$\mathrm{Aut}(\mathcal{C}) \cong G(\mathbb{R})$ is $\mathbb{R}$-points of a reductive group G.*

2. *if $\Gamma \leq G$ is an arithmetic subgroup acting on $\mathcal{C}$, then there is a rational polyhedral fundamental domain Δ for $\mathcal{C}^+$ under the action of Γ.*

Sketch of proof theorem(1.4)

First we have a group homomorphism

$$\mathrm{Aut}(X) \to O(\mathrm{NS}(X)), \quad f \ \mapsto f^*$$

which gives the action of $\mathrm{Aut}(X)$ on cones of K3 X, whose image is a finite index subgroup.

Second, Observe that as

$$\mathrm{Pos}(X) \cong \{x \in \mathrm{NS}(X)_{\mathbb{R}} \mid x_0^2 > x_1^2 + \cdots + x_{\rho-1}^2\}^+$$

where $x_0, \cdots, x_{\rho-1}$ is a $\mathbb{R}$-basis for the real vector space $\mathrm{NS}(X)_{\mathbb{R}}$. Thus, $\mathrm{Aut}(\mathrm{Pos}(X)) = G(\mathbb{R})$ where $G \leq O(\mathrm{NS}(X))$. So it is easy to check that $\mathrm{Pos}(X)$ is a homogeneous self-dual cone. Now apply the criterion (1.6) for the cone $\mathcal{C} = \mathrm{Pos}(X) \subset \mathrm{NS}(X)_{\mathbb{R}}$, we can get a rational polyhedral cone Δ for the action of $O(\mathrm{NS}(X))$ on $\mathrm{Pos}(X)^+$, where $\mathrm{Pos}(X)^+$ is the cone spanned by effective positive classes since $\mathrm{Pos}(X)^+$ is the convex hull of $\overline{\mathrm{Pos}(X)} \cap \mathrm{NS}(X)_{\mathbb{Q}}$.

The last step is to use the Weyl group

$$W := \langle s_\delta : \mathrm{NS}(X) \to \mathrm{NS}(X) \mid \delta \in \mathrm{NS}, \ \delta^2 = -2\rangle$$

to translate Δ inside $\mathrm{Nef}^e(X)$.

Remark 1.7. *For some K3, we have $\mathrm{Nef}(X) = \overline{\mathrm{Pos}(X)}$, in this case the proof will be finished after applying the criterion (1.6). But this is not always true.*

Remark 1.8. *The similar structure for cones of Calabi-Yau varieties in higher dimension are conjectured by Morrison (see [7] [6]) and Kawamata (see [3]) respectively from different perspective. But for higher dimensional Calabi-Yau varieties, the cone conjecture is only known very special cases, e.g., for hyperkaler variety, due to Ekaterina-Verbisky [1].*

Remark 1.9. *According to the minimal model theory, if the Cone conjecture holds for $\mathrm{Nef}^e(X)$ under action of $\mathrm{Aut}(X)$, it will imply that there are only finite contraction $X \to Y$ up to $\mathrm{Aut}(X)$. This is also one of the motivations for the cone conjecture.*

2 Moduli spaces of K3 surfaces

2.1 Crash course on DM stacks

Let $\mathcal{S}\mathrm{ch}(S)$ a the étale site (i.e., a category with Grothendick topology whose covering is given by étale morphism of schemes) and

$$p: \ \mathfrak{X} \to \mathcal{S}\mathrm{ch}(S)$$

is a category fibered in groupoid over $\mathcal{S}\mathrm{ch}(S)$, that is,

1. (existence of pullback) for any morphism $\phi : X \to Y \in \mathrm{Mor}_{\mathcal{S}\mathrm{ch}(S)}(X, Y)$ and $w \in p^{-1}(Y)$, there is a $v \in \mathfrak{X}$ and morphism $f \in \mathrm{Mor}_{\mathfrak{X}}(v, w)$ such that $p(\phi) = f$.
2. (universal properties of pullback)

Definition 2.1. *$p: \ \mathfrak{X} \to \mathcal{S}\mathrm{ch}(S)$ is a stack if the following gluing axioms hold*

1. *(Gluing objects) for $X \in \mathcal{S}\mathrm{ch}(S)$ and $v, w \in p^{-1}(X)$, the functor $p^{-1}(X) \to \mathcal{S}\mathrm{et}$ defined by*

$$(Y \xrightarrow{f} X) \ \mapsto \ \mathrm{Mor}_{p^{-1}(Y)}(\widetilde{f}(v), \widetilde{f}(w))$$

is a sheaf where $\widetilde{f}$ is the pullback of f, that is, for any covering $\{X_i \to X\}$ for $X \in \mathcal{S}\mathrm{ch}(S)$ and any objects v_i over X_i with isomorphism

$$\phi_{ij} : \ v_i|_{X_{ij}} \to v_j|_{X_{ij}}$$

satisfying the cocycle condition $\phi_{ij}|_{X_{ijk}} \circ \phi_{jk}|_{X_{ijk}} = \phi_{ik}|_{X_{ijk}}$, then there are objects v over X with isomorphism $f_i : v|_{X_i} \to v_i$ such that

$$\phi_{ij} \circ f_i|_{X_{ij}} = f_j|_{X_{ij}}.$$

In other words, there is exact

$$\mathfrak{X}(X) \to \prod_i \mathfrak{X}(X_i) \rightrightarrows \prod_{ij} \mathfrak{X}(X_{ij}) \Rrightarrow \prod_{ijk} \mathfrak{X}(X_{ijk})$$

2. *(Gluing morphisms) For any $v, w \in p^{-1}(X)$ with morphism $\phi_i : \ v|_{X_i} \to w$ such that*

$$\phi_i|_{X_{ij}} = \phi_j|_{X_{ij}},$$

then there is a unique morphism $\phi : v \to w$ such that $\phi|_{X_i} = \phi_i$. In other words, the isom presheaf [2]

$$\mathrm{Isom}(v, w) : \ p^{-1}(X) \to \mathcal{S}\mathrm{et}, \ Y \xrightarrow{f} X \ \mapsto \ \mathrm{Mor}_{p^{-1}(Y)}(f^*v, f^*w)$$

is a sheaf.

If in addition, the stack $\mathfrak{X} \to \mathcal{S}\mathrm{ch}(S)$ satisfies

- *The diagonal morphism of stack*

$$\Delta : \ \mathfrak{X} \to \mathfrak{X} \times_{\mathcal{S}\mathrm{ch}(S)} \mathfrak{X}$$

is representable, qusai-compact and separated.

- *There is a scheme X with an étale surjective morphism*

$$X \to \mathfrak{X},$$

then we call $\mathfrak{X}$ a Deligne-Mumford (DM) stack.

Theorem 2.2 (Keel-Mori)**.** *There is a coarse moduli space M for a DM stack $\mathfrak{M} \to \mathcal{S}\mathrm{ch}(S)$.*

[2]Note that here $Y \xrightarrow{f} X$ is a S-morphism in $\mathcal{S}\mathrm{ch}(S)$.

2.2 Moduli problem for K3 surfaces

We consider the moduli problem $\mathfrak{F}_g^\circ: \ \mathcal{S}\mathrm{ch}(\mathbb{C})^{op} \longrightarrow \mathcal{S}\mathrm{et}$ defined by

$$\mathfrak{F}_g^\circ(B) := \{(\mathfrak{X},\mathcal{L}) \xrightarrow{\pi} B \mid \pi \text{ smooth, proper}, \mathcal{L}\ \pi\text{-ample, primitive}, \ \mathcal{L}_b^2 = 2g-2\}/\sim \tag{2}$$

where $(\mathfrak{X},\mathcal{L}) \xrightarrow{\pi} B \sim (\mathfrak{X}',\mathcal{L}') \xrightarrow{\pi'} B$ means there is an isomorphism $f: \mathfrak{X} \to \mathfrak{X}'$ such that

$$f^*\mathcal{L}' = \mathcal{L} \otimes \pi^* A, \text{ for some } A \in \mathrm{Pic}(B).$$

More generally, we can consider lattice polarised K3 surface

$$\mathfrak{F}_\Lambda: \ \mathcal{S}\mathrm{ch}(\mathbb{C})^{op} \longrightarrow \mathcal{S}\mathrm{et}$$

for any sublattice $\Lambda \subset H^2(K3,\mathbb{Z}) = U^3 \bigoplus E_8^2(-1)$ of signature $(1,m)$.

Definition 2.3. *A moduli functor $\mathfrak{M} \to \mathcal{S}\mathrm{et}$ is said to*

1. *fine representable by M if there is a scheme M such that $\mathfrak{M} = h_M$ where*

$$h_M: \mathcal{S}\mathrm{ch}(\mathbb{C})^{op} \to \mathcal{S}\mathrm{et}, \quad h_M(B) = \mathrm{Hom}(B,M).$$

2. *coarsely representable by M if there is a scheme M such that there is a natural transformation $\eta: \mathfrak{M} \to h_M$ such that there is a bijection $M(\mathrm{Spec}(\mathbb{C})) \cong \mathfrak{M}(\mathrm{Spec}(\mathbb{C})$ and for any scheme N and morphism $\phi: \mathfrak{M} \to h_N$, there is unique scheme morphism $\psi: M \to N$ such that the following diagram*

$$\begin{tikzcd} \mathfrak{M} \arrow[r, "\eta"] \arrow[d, "\forall\ \phi"'] & h_M \arrow[dl, "\exists\ \psi"] \\ h_N & \end{tikzcd}$$

commutes.

In both cases, there is a one to one correspondence $M(\mathrm{Spec}(\mathbb{C})) \cong \mathfrak{M}(\mathrm{Spec}(\mathbb{C})$.

Remark 2.4. *One may allow the family π has singular fiber with ADE singularities at worst, in this way we get a moduli functor $\mathfrak{F}_g$ which is also separated. Meanwhile, one may also allow that L is just big and nef to get a new moduli functor $\mathfrak{F}_g'$, but as a stack $\mathfrak{F}_g'$ is not separated. The reason is that there are two families $(\mathfrak{X},\mathcal{L})$ and $(\mathfrak{X}',\mathcal{L}')$ over $\mathrm{Spec}(R)$ for a DVR R where both generic fiber and special fiber are isomorphic, but the isomorphism can not extend to the whole family. But both $\mathfrak{F}_g$ and $\mathfrak{F}_g'$ are coarsely representable and their coarse moduli spaces are isomorphic.*

Remark 2.5. *In AG, the fine moduli problems are rare. The most important two examples are Hilbert scheme (used to construct moduli space of various kinds of varieties) and Quot scheme (used to construct moduli space of vector bundles or sheaves on a variety).*

The moduli functor can be viewed as a category

$$\mathfrak{F}_g \to \mathcal{S}\mathrm{ch}(S), \quad [(\mathfrak{X},\mathcal{L}) \xrightarrow{\pi} B] \ \mapsto \ B$$

fibered over $\mathcal{S}\mathrm{ch}(S)$, each fiber $\mathfrak{F}_g(B)$ is a groupoid. That is, $\mathfrak{F}_g$ is a category fibered in groupoid. Then one can show $\mathfrak{F}_g$ is an algebraic stack by checking the axioms of algebraic stack and use abstract moduli stack theory to show $\mathfrak{F}_g$ can be coarsely represented by an algebraic space.

Lemma 2.6 ([8] Theorem8.3, Boundedness of polarised K3)**.** *If L is an ample line bundle on a K3 surface X, then L^3 is very ample.*

Proof. We just sketch the idea of the proof. Recall for any line bundle L with $h^0(L) > 1$, we have a rational map given by the line bundle

$$\phi_L: \ X \dashrightarrow \mathbb{P}^{h^0(L)-1} = \mathbb{P}H^0(L)^\vee$$

which is defined over $X - Bs(|L|) = X - \bigcap_{s \in H^0(L)} \{s = 0\}$.

To show the line bundle L^3 giving a closed embedding into $\mathbb{P}H^0(L^3)^\vee$, it is equivalent to show L^3 separates points and tangents (See Hartshone chapter 2, Proposition 7.3). i.e., we need to show

1. (separating points) for any $x \neq y \in X$ closed points,

$$H^0(L^3) \to H^0(L^3_x) \oplus H^0(L^3_y) \to 0.$$

   By the exact sequence $0 \to \mathfrak{m}_x \otimes \mathfrak{m}_y \to \mathcal{O}_X \to \mathcal{O}_x \oplus \mathcal{O}_y \to 0$, it is sufficient to show

$$H^1(L^3 \otimes \mathfrak{m}_x \otimes \mathfrak{m}_y) = 0. \tag{3}$$

2. (separating tangents) for any $x \in X$ closed point,

$$H^0(L^3 \otimes \mathfrak{m}_x) \to H^0(L^3 \otimes \mathfrak{m}_x/\mathfrak{m}_x^2) \to 0.$$

   Similar arguments reduce this to show

$$H^1(L^3 \otimes \mathfrak{m}_x^2) = 0. \tag{4}$$

Note that for the blowup $\pi : Y := \ Bl_p(X) \to X$, the direct image of sheaf

$$R^m\pi_*\mathcal{O}_{Bl_p(X)}(-lE) = \begin{cases} 0, & m > 0, \\ \mathfrak{m}_x^l, & m = 0. \end{cases}$$

where E is the exceptional divisor and $l \geq 0$. Thus by the Leray spectral sequence for π and projection formula

$$R^m\pi^*L^3 \otimes \mathcal{O}(-2E) = L^3 \otimes R^m\pi^*\mathcal{O}(-2E),$$

the vanishing results (3) and (4) follow from the following vanishing results of the numerically 1-connected divisors.

We explain the vanishing (4) as an example. Recall an effective divisor D on a smooth projective surface X is called numerically m-connected if for any decomposition $D = D_1 + D_2$ of D into two nonzero effective divisors D_1, D_2, then

$$D_1.D_2 \geq m.$$

We have the following basic facts:

1. C.P. Ramanujam's lemma: If D is a numerically 1-connected divisor, then $h^0(D, \mathcal{O}_D) = 1$.Thus if $H^1(X, \mathcal{O}_X) = 0$, then $H^1(X, \mathcal{O}_X(-D)) = 0$.

   The second assertion follows from the long exact sequence obtained by taking cohomology of

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0.$$

2. Let $\pi : \ Bl_p(X) \to X$ be the blowup with exceptional divisor E. If $C \subset X$ is a curve so that each divisor in $|\mathcal{O}_X(C)|$ is numerically 2-connected, then each divisor in $|\mathcal{O}_{Bl_pX}(\pi^*C - 2E)|$ is numerically 1-connected.

3. If X is a K3 surface and C is irreducible curve so that $C^2 > 0$, then $|\mathcal{O}_X(C)|$ is numerically 2-connected.

4. If X is a K3 surface and L ample, then there is an irreducible curve in $|L^3|$.

The above result 2,3 and 4 are due to Saint-Donat's work ([8]) on detailed analysis of linear system of a line bundle L on a K3 surface. Now by taking an irreducible curve $C \in |L^3|$, then C is 2-connected and thus $\pi^*C - 2E$ is 1-connected. By Ramanujam's lemma,

$$H^1(X, \pi^*L^3 \otimes \mathcal{O}(-2E)) = 0.$$

This implies (4). ☐

By the boundedness result, we may consider the Hilbert scheme, which parametrizes closed subschemes in a fixed ambient space with fixed Hilbert polynomials. In our case, we need

$$\mathcal{H}\mathrm{ilb}_p(\mathbb{P}^N) : \mathcal{S}\mathrm{ch}(\mathbb{C})^{op} \to \mathcal{S}\mathrm{et}$$

defined by

$$T \mapsto \{Z \subset T \times \mathbb{P}^N \to T \text{ flat over } T\}$$

where the Hilbert polynomial $p(t) = \chi(3tL)$ as we can embed each polarised K3 (X, L) into $\mathbb{P}H^0(L^3) \cong \mathbb{P}^N$ where

$$N = \frac{(3L)^2}{2} + \chi(\mathcal{O}_X) - 1 = 9l^2 + 1.$$

By the results of Grothendieck, the functor $\mathcal{H}\mathrm{ilb}_p(\mathbb{P}^N)$ defines a fine moduli problem and admits a fine moduli space $\mathrm{Hilb}_p(P^N)$. Indeed, the ideal sheaves of closed subscheme in $\mathbb{P}^N$ with fixed Hilbert polynomial have a uniform boundedness result: there is a $m = m(p) \in \mathbb{N}$ such that for any closed subscheme $X \subset P^N$ with Hilbert polynomial

$$p(l) = \chi(X, \mathcal{O}_{\mathbb{P}^N}|_X(l)) \ \in \mathbb{Q}[l],$$

the cohomology $H^1(X, \mathcal{I}_X(m)) = 0$, which implies such X can be reconstructed from

$$0 \to \mathcal{I}_X \to \mathcal{O}_{\mathbb{P}^N} \to \mathcal{O}_X \to 0$$

after twisting $\mathcal{O}_{\mathbb{P}^N}(m)$. [3] In this way, $\mathcal{H}\mathrm{ilb}_p(\mathbb{P}^N)$ can be viewed as a closed sub functor of Grassmannian functor

$$\mathcal{G}\mathrm{rass}(p(m), H^0(\mathcal{O}_{\mathbb{P}^N}(m)) : \mathcal{S}\mathrm{ch}(\mathbb{C})^{op} \to \mathcal{S}\mathrm{et}$$

$\mathcal{G}\mathrm{rass}(p(m), H^0(\mathcal{O}_{\mathbb{P}^N}(m))$ has a fine moduli space $\mathrm{Grass}(p(m), H^0(\mathcal{O}_{\mathbb{P}^N}(m))$. This also implies $\mathcal{H}\mathrm{ilb}_p(\mathbb{P}^N)$ has a fine moduli space $\mathrm{Hilb}_p(P^N)$ after restricting to the closed locus.

Now we take $U \subset \mathrm{Hilb}_p(P^N)$ the locus parametrizing polarised K3 surface (X, L). [4] Note that there is a natural group $\mathrm{PGL}(N+1)$ acting on $\mathrm{Hilb}_p(P^N)$ via changing the coordinate of $\mathbb{P}^N$. Clearly, U is a $\mathrm{PGL}(N+1)$ invariant locus. Therefore, we get a quotient stack

$$[U/\,\mathrm{PGL}(N+1)] : \ \mathcal{S}\mathrm{ch}(\mathbb{C})^{op} \to \mathcal{S}\mathrm{et}$$

A very basic results is the representability of quotient stack

Theorem 2.7. *Let X be a scheme of finite type and G be a reductive group acting properly and linearly on X, then the quotient stack $[X/G]$ admits a coarse moduli space X/G.*

Here the proper action is equivalent to say

1. the orbit $G \cdot x \subset X$ is closed in X.
2. the stabilizer G_x of point x is finite.

Now we are going to show

Theorem 2.8. *There are isomorphism of the two moduli stack $\mathfrak{F}_g^\circ \cong [U/\,\mathrm{PGL}(N+1)]$.*

Proof. It is sufficient to show the morphism $[U/\,\mathrm{PGL}(N+1)] \to \mathfrak{F}_g^\circ$ will define a injective morphism. This is done by showing for any two isomorphic families $(\mathfrak{X}, \mathcal{L}) \xrightarrow{\pi} B$ and $(\mathfrak{X}', \mathcal{L}') \xrightarrow{\pi'} B$ differs by action of $\mathrm{PGL}(N+1)$. Indeed, isomorphism says there is an isomorphism $f : \mathfrak{X} \to \mathfrak{X}'$ such that

$$f^*\mathcal{L}' = \mathcal{L} \otimes \pi^* A, \text{ for some } A \in \mathrm{Pic}(B)$$

Then projection formula shows

$$\pi_*(\mathcal{L}^m) = \pi'_* \circ f_*(f^*\mathcal{L}'^m \otimes \pi^* A^{-m}) = A^{-m} \otimes \pi'_*(\mathcal{L}'^m)$$

from the commutative diagram

$$\begin{tikzcd}
(\mathfrak{X}, \mathcal{L}) \arrow[rr] \arrow[dr, "\pi"'] & & (\mathfrak{X}', \mathcal{L}') \arrow[dl, "\pi'"] \\
& B &
\end{tikzcd}.$$

The natural isomorphism $\mathrm{Proj}_{\mathcal{O}_B}(\pi_*(\mathcal{L}^m)) = \mathrm{Proj}_{\mathcal{O}_B}(A^{-m} \otimes \pi'_*(\mathcal{L}'^m)) \cong \mathrm{Proj}_{\mathcal{O}_B}(\pi'_*(\mathcal{L}'^m))$ differs by $\mathrm{PGL}(N_m)$. □

[3] More precisely, X has coordinate ring $\mathrm{Sym}^*(H^0(\mathcal{O}_{\mathbb{P}^N}(m)))/I_X$ where I_X is the ideal of symmetric algebra $\mathrm{Sym}^*(H^0(\mathcal{O}_{\mathbb{P}^N}(m)))$ generated by $H^0(\mathcal{I}_X(m))$.

[4] $U \subset \mathrm{Hilb}_p(P^N)$ is locally closed as it paremattrizes smooth object in the family $\mathcal{U} \to \mathrm{Hilb}_p(P^N)$.

As a corollary, to show the moduli stack $\mathfrak{F}_g^\circ$ admits a coarse moduli space, we just need to check the action $\mathrm{PGL}(N+1)$ on U is proper. As the proper action is also equivalent to the proper morphism

$$\mathrm{PGL}(N+1) \times U \to U \times U, \quad (g, x) \ \mapsto \ (g \cdot x, x)$$

it can also be obtained by the following result

Lemma 2.9 (Matsusaka-Mumford [5]). *Let R be a DVR, K the fractional field and $k = R/\mathfrak{m}$ the residue filed. Assume X, Y are smooth projective variety over K and $T \subset X \times_K Y$ the graph of an isomorphism. $D_X \subset X$ and $D_Y \subset$ are smooth divisors. Denote $(\mathfrak{X}_k, D_{\mathfrak{X}_k}), (\mathcal{Y}_k, D_{\mathcal{Y}_k})$ and $\mathcal{T}_k$ the reductions via R to k such that $\mathfrak{X}_k, \mathcal{Y}_k$ are nonsingular and $D_{\mathfrak{X}_k}$ is non-degenerate on $\mathfrak{X}_k$(repsectively for $D_{\mathcal{Y}_k}$). If either $\mathfrak{X}_k$ or $\mathcal{Y}_k$ is not uniruled, then $\mathcal{T}_k$ is the graph of an isomorphism between $\mathfrak{X}_k$ and $\mathcal{Y}_k$.*

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